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LETTER TO THE EDITOR

Hierarchy-induced isotropy–anisosotropy transition on a fractal resistor network

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Abstract. A Sierpinski gasket resistor network is considered in which the basic microscopically anisotropic resistance distribution has a hierarchical pattern. The system is shown to undergo a transition from macroscopic isotropy to macroscopic anisotropy as a hierarchical parameter R is varied. For $R < R_c$, the system may restore macroscopic isotropy, regardless of the degree of anisotropy on the microscopic scale. For $R > R_c$, on the other hand, the system remains anisotropic on the macroscopic scale due to the basic anisotropy on the microscopic scale. The degree of macroscopic anisotropy depends on both the hierarchical parameter R and the microscopic anisotropy.

Recently, Barlow and co-workers [1] reported a new type of restoration of macroscopic isotropy in fractal systems with microscopic anisotropy. The phenomenon is unique in the sense that it is absent in uniform media such as regular lattices or Euclidean spaces, while it is claimed to be universal since it can be observed in many physical setups on a wide class of fractal systems [1]. In the present study, we address the question whether or not such a type of restoration of macroscopic isotropy in fractal systems with microscopic anisotropy can be observed if the fractal possesses hierarchical structure.

The physical properties of hierarchical structures have attracted much attention in recent years, since these structures are believed to arise in various physical contexts (see [2] for a review). Even in one-dimensional (1D) systems, studies concerning the hierarchical structures have revealed a wealth of interesting features. For instance, in the case of diffusion, it has been shown that a hierarchical system can undergo a phase transition from anomalous to ordinary diffusion, as well as display anomalous diffusion behaviour, when the hierarchical parameter R is varied. Here R is a positive paratmeter describing the hierarchy of physical quantities such as transition rates, interactions and resistances (see, e.g., [3]). In the case of directly measurable quantities, the study on AC hopping conductivity of a 1D hierarchical system showed that the conductivity can have rather different types of low- and high-frequency behaviour depending on the value of R [4]. For fractal systems, on the other hand, the hierarchical structure can be provided, more naturally, by the fractal nature of some fractals such as the Sierpinski gasket and regular Vicsek fractals. A study of the relaxation problem in a fractal system with a hierarchical array of barriers also indicated that a transition from anomalous to normal diffusion occurs as R varies [5]. More recently, it was found that all non-degenerate transverse vibrational modes on a regular Vicsek fractal with a hierarchical pattern of nearest-neighbour interactions may exhibit non-decay (extended) or power-law decay spatial scaling behaviour depending on the value of R [6]. Now, a natural question is whether or not the new type of restoration of macroscopic isotropy in fractal systems with basic (microscopic) anisotropy can be observed in the fractals with hierarchical structure. To answer this question, we consider, as an example, a Sierpinski gasket resistor network with microscopic anisotropy and a hierarchical pattern in resistance distribution. A transition from macroscopic isotropy to macroscopic anisotropy is found as the hierarchical parameter R is varied. For $R < R_c$, the system may restore macroscopic isotropy, regardless of the degree of anisotropy on the microscopic scale, while for $R > R_c$, the system stays anisotropic on the macroscopic scale due to the basic microscopic anisotropy, and the degree of macroscopic anisotropy is governed by the hierarchical parameter R as well as the microscopic anisotropy.

The model treated here is similar to that of [5] in studying the diffusion problem; the anisotropy on the basic microscopic scale is introduced in a way analogous to that of [1]. The zero-stage Sierpinski gasket resistor network is a triangle. We associate a resistor of resistance 1 with the bond in the horizontal direction and a resistor of resistance r with each of the remaining two bonds of the triangle (see figure 1(a)). Here $r \neq 1$ parametrizes the basic microscopic anisotropy. The hierarchy for the fractal resistor network is introduced in the first- and the higher-stage fractals. In the present model, a first-stage Sierpinski gasket network is constructed by assembling three copies of the zero-stage network at the three corners of a regular triangle (see figure 1(b)). However, in contrast to the model studied by Barlow and co-workers [1], in our fractal network, there exists resistance for each link connecting any two adjoining zero-stage fractal networks. The hierarchy and anisotropy are introduced by setting the resistance of the link in the horizontal direction equal to R, while each of the remaining two links connecting the zero-stage fractal networks is associated with resistance Rr. The higher-stage fractal network is built by a similar aggregation pattern. That is, an $(n+1)$th-stage fractal network is made by assembling three copies of the nth-stage fractal networks at three corners of a regular triangle. Every two nth-stage fractal networks are connected by a link with resistance of $R^{(n+1)}$ ($R^{(n+1)}r$) if the connecting link is along the horizontal (other) direction(s). See figure 1(c) for a second-stage fractal network constructed from three first-stage networks. Note that if one sets $R = 0$, the present model reduces to that studied in [1].

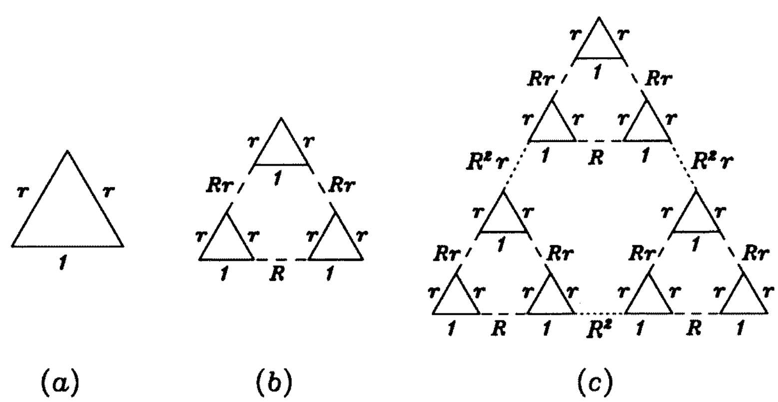

Figure 1. Construction of a Sierpinski gasket resistor network by aggregation: (a) stage 0, (b) stage 1 and (c) stage 2. In the nth-stage network, every two $(n-1)$th-stage networks are connected by a link with resistance R^n (horizontal direction) or $R^n r$ (other directions).

By repeated use of the star–triangle transformation relations, any nth-stage fractal network can be transformed into a simple equivalent triangle network. Let X_n denote the resistance for the bond in the horizontal direction of such a triangle network obtained from an nth-stage one, and Y_n be that for the bond in other directions. It follows from the definition that

$$X_0 = 1 \qquad Y_0 = r\,. \tag{1}$$

Define $H_n = X_n/Y_n$, the logarithm of which, $\ln H_n$, is used to measure the degree of anisotropy of the nth-stage fractal network composed of resistance elements with basic microscopic anisotropy parametrized by $H_0 = X_0/Y_0 = 1/r$. From the self-similarity of the fractal structure, one can easily derive the recursion equations for Y_n and H_n, by using the star–triangle relations, as follows:

$$Y_{n+1} = \frac{(2H_n+3)Y_n}{(H_n+2)} + R^{n+1}r \tag{2}$$

$$H_{n+1} = \frac{(6H_n^2+4H_n)Y_n^2 + (2+H_n)(1+2H_n+4rH_n)R^{n+1}Y_n + (2+H_n)^2 r R^{2(n+1)}}{(H_n^2+6H_n+3)Y_n^2 + (2+H_n)(1+4r+2rH_n)R^{n+1}Y_n + (2+H_n)^2 r^2 R^{2(n+1)}} \tag{3}$$

while X_{n+1} is readily given by $X_{n+1} = H_{n+1}Y_{n+1}$. Starting from the initial conditions (1), by repeatedly using the recursion equations, one can find that H_n goes to a stable fixed point, the value of which depends on the value of R as described below.

(i) For $0 < R < R_c = \frac{5}{3}$, the first term on the right-hand side of (2) is dominant. It is thus not difficult to show [1] that

$$\lim_{n\to\infty} H_n = 1 \tag{4}$$

with the rate of restoration of isotropy $H_{n+1} - 1 \approx \frac{4}{5}(H_n - 1)$ and X_n and Y_n satisfying the scaling behaviour

$$X_{n+1} \approx Y_{n+1} \approx \tfrac{5}{3}X_n \tag{5}$$

for large n. The macroscopic isotropy is restored independent of the value of microscopic anisotropy measured by r as in the system with $R = 0$ [1]. Figure 2 gives the calculated behaviour of $\ln H_n$ for systems with two typical values of $R < R_c$ and different values of r, from which the restoration of macroscopic isotropy is clearly observed regardless of the basic anisotropy r.

(ii) For $R \geqslant R_c = \frac{5}{3}$, one has

$$\lim_{n\to\infty} H_n = \frac{\sqrt{\Delta} - (R-2)(2r-1)}{2(R-1)r} \equiv h \tag{6}$$

with

$$\Delta = (R-2)^2(1+2r)^2 + 4r(3R-5) \tag{7}$$

and

$$\frac{2h+3}{h+2} \leqslant R\,. \tag{8}$$

The rate of approach to the fixed point h is given by $H_{n+1} - h \approx \eta(H_n - h)$, where η, a function of R and r, is smaller than $\frac{4}{5}$, suggesting that h should be a stable fixed point of

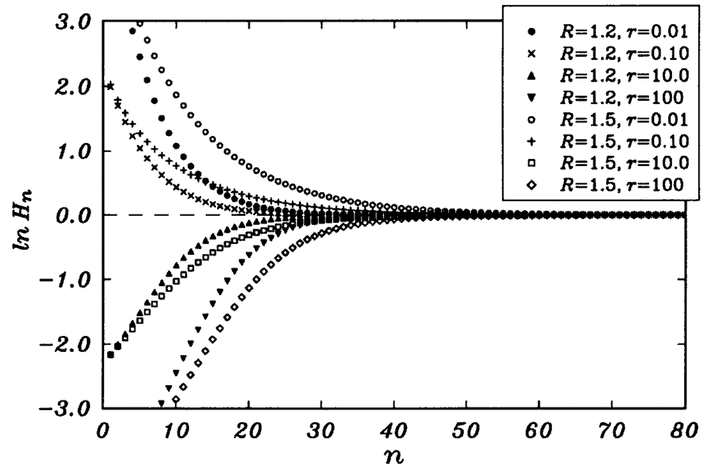

Figure 2. $\ln H_n$ as a function of stage number n for the fractal resistor network with two typical values of $R < R_c = \frac{5}{3}$ and different values of r. Restoration of macroscopic isotropy is clearly observed regardless of the value of r.

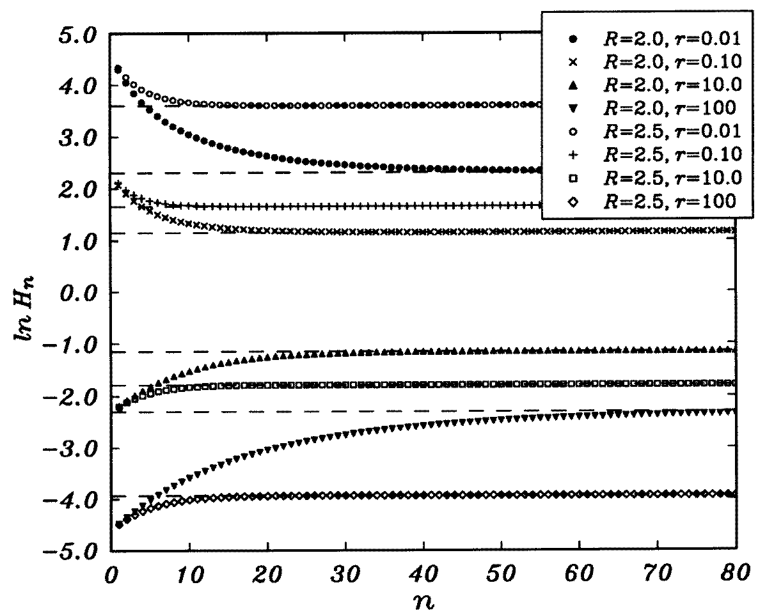

Figure 3. $\ln H_n$ as a function of stage number n for the fractal resistor network with two typical values of $R > R_c$ and different values of r. It can be seen that the system remains anisotropic on the macroscopic scale due to the basic anisotropy $r \neq 1$, while the degree of macroscopic anisotropy, measured by $\ln h$ and shown by the horizontal dashed lines in the figure, depends on both R and r.

the recursion relations (2) and (3). As a result, one obtains the following asymptotic scaling behaviour for X_n and Y_n:

$$X_n \approx hY_n \approx \frac{h(h+2)rR^{n+1}}{(h+2)R-(2h+3)} \,. \tag{9}$$

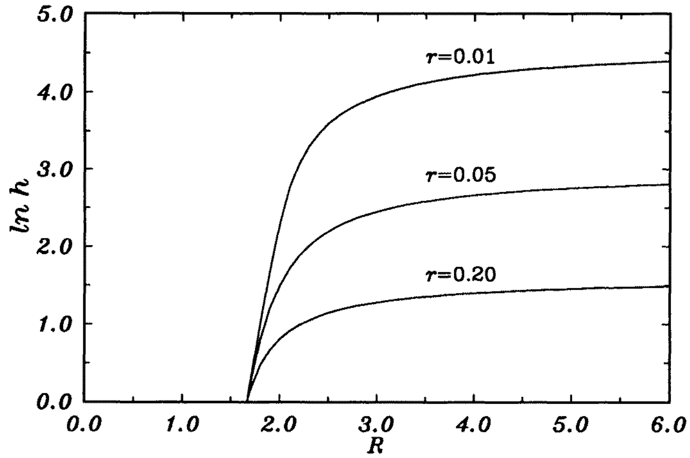

Figure 4. The order parameter, $\xi \equiv |\frac{h-1}{h+1}|$, as a function of the hierarchical parameter R for some different values of r. An isotropy–anisosotropy transition is observed at $R = R_c = \frac{5}{3}$.

It can be seen that the degree of macroscopic anisotropy, measured by $\ln h$, as well as the rate of approach to the fixed point, depends on both the hierarchical parameter R and the basic anisotropy r. Figure 3 shows the calculated behaviour of $\ln H_n$ for systems with two typical values of $R > R_c$ and different values of r, from which one can find an apparent r and R dependence of the macroscopic anisotropy $\ln h$.

It is therefore concluded that the fractal network undergoes a hierarchy-induced transition from macroscopic isotropy to macroscopic anisotropy as the hierarchical parameter R is varied. Define $\xi \equiv |\frac{h-1}{h+1}|$ as an order parameter, which is normalized in such a way that its value ranges from 0 to 1 for all R and r. Figure 4 displays the order parameter ξ as a function of R for some values of $r < 1$ (the cases with $r > 1$ are similar). An isotropy–anisosotropy transition is observed at $R = R_c = \frac{5}{3}$. In addition, it follows from (6) that

$$\xi \equiv \left|\frac{h-1}{h+1}\right| \approx \frac{9|1-r|}{2(1+2r)}(R - R_c) \qquad \text{as } R \to R_c^+ \tag{10}$$

implying that the isotropy–anisosotropy transition should be of second order.

In summary, we have studied a Sierpinski gasket resistor network which has hierarchical and microscopically anisotropic resistance distribution. A transition from macroscopic isotropy to macroscopic anisotropy is found as the hierarchical parameter R is increased. For $R < R_c = \frac{5}{3}$, the system may restore macroscopic isotropy independently of the degree of basic anisotropy r on the microscopic scale, while for $R > R_c$, the recovery of isotropy on the macroscopic scale can no longer be observed, and the degree of macroscopic anisotropy depends on the hierarchical parameter R as well as the microscopic anisotropy r. It is suspected that this transition may be observed experimentally, for example, in some transport phenomena in fractal media, where the hierarchical parameter R may be temperature dependent.

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